Effect of sigma meson on the $D_1(2430) \to D\pi\pi$ decay

Masayasu Harada, Hironori Hoshino, and Yong-Liang Ma

Department of Physics, Nagoya University, Nagoya, 464-8602, Japan.

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We study the effect of sigma meson on the $D_1(2430) \to D\pi\pi$ decay by constructing an effective Lagrangian preserving the chiral symmetry and the heavy quark symmetry. The sigma meson is included through a linear sigma model, in which both the $q\bar{q}$ and $qq\bar{q}\bar{q}$ states are incorporated respecting their different $U(1)_A$ transformation properties. We first fit the sigma meson mass and σ-π-π coupling constant to the $I = 0$, S-wave π-π scattering data. Then, we show how the differential decay width $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$ depends on the quark structure of the sigma meson. We find that our study, combing with the future data, can give a clue to understand the sigma meson structure.

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I. INTRODUCTION

The lightest scalar meson "sigma" is an interesting object which may give a clue to understand some fundamental problems of QCD such as the chiral symmetry structure, the origin of mass and so on. The mass spectrum of the light scalar meson nonet including the sigma meson disfavors the $q\bar{q}$ picture but prefers the $qq\bar{q}\bar{q}$ interpretation (see, e.g. Ref. [1, 2] and references therein).

If in the nature there are both $q\bar{q}$ and $qq\bar{q}\bar{q}$ scalar states, they might mix to give the physical scalar mesons (sometimes the glueball component is included). In the literature, this superposition has been discussed widely [3], but the structure of the light scalar mesons remains an open question and deserves further investigation.

To investigate the quark contents of the light scalar mesons, we should find some quantities that can distinguish different components, two-quark component, four-quark component and glueball. In Ref. [4] it was pointed that, in a linear sigma model expressing the spontaneous chiral symmetry breaking, the $q\bar{q}$ and $qq\bar{q}\bar{q}$ bound states have different charges of $U(1)_A$ symmetry therefore one can use this $U(1)_A$ transformation property to discriminate the $q\bar{q}$ and $qq\bar{q}\bar{q}$ components of the scalar mesons. The $U(1)_A$ symmetry is explicitly broken by anomaly in QCD which causes a mixing between the $q\bar{q}$ states and the $qq\bar{q}\bar{q}$ states [5–7]. This contribution is included in the model Lagrangian in such a way that the anomaly matching condition is satisfied. Then, unlike the other terms which are invariant under the $U(1)_A$ transformation, the $U(1)_A$ violating terms are constrained by the anomaly. Another source of the mixing between $q\bar{q}$ and $qq\bar{q}\bar{q}$ comes from the existence of both the $q\bar{q}$ and $qq\bar{q}\bar{q}$ condensation [5, 7].

In this paper, we devote ourselves to study the sigma meson structure in the heavy-light meson decay, specifically, the $D_1(2430) \to D(\pi\pi)_{I=0,L=0}$ decay, based on the chiral partner structure between (D^*, D) and $(D_1(2430), D_0^*(2400))$ [8, 9] (in the following, we simply denote $D_1(2430)$ and $D_0^*(2400)$ as D_1 and D_0^*, respectively). An interesting property of this process is that there is only one light quark in the heavy-light meson so that the axial transformation property is well determined and the heavy-light meson couples only to the two-quark component of the scalar meson.

The paper is organized as follows: In sec. II, we introduce an extended linear sigma model for three flavor QCD. Section III is devoted to study the π-π scattering: We determine the σ-π-π coupling and sigma meson mass by fitting them to the data for the $I = 0$, S-wave channel. In sec. IV, we construct an effective Lagrangian for the interaction among light mesons and heavy-light mesons. In sec. V, after determining the relevant parameters in the heavy-light meson sector, we show how the differential decay width $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$ depend on the quark structure of the sigma meson. We give a summary and discussions in sec. VI. In appendices we show some details of derivations of several formulas.

II. LINEAR SIGMA MODEL WITH TWO-QUARK AND FOUR-QUARK STATES

In this section, we introduce a linear sigma model for three flavor QCD in the low-energy region by including a 3×3 chiral nonet field M representing the $q\bar{q}$ states and a 3×3 chiral nonet field M' standing for the $qq\bar{q}\bar{q}$ states. These two nonets have the same chiral $SU(3)_L \times SU(3)_R$ transformation property:

$$M \to g_L M g_R^\dagger , \quad M' \to g_L M' g_R^\dagger , \tag{1}$$

where $g_{L,R} \in SU(3)_{L,R}$. On the other hand, as pointed in Ref. [4, 5], they have the following different $U(1)_A$ transformation properties:

$$M \to e^{2i\alpha} M , \quad M' \to e^{-4i\alpha} M', \tag{2}$$

with α as the phase factor of the axial transformation. We decompose M and M' as

$$M = S + i\Phi , \quad M' = S' + i\Phi' , \tag{3}$$

where S and S' are the scalar meson matrices and Φ and Φ' are the pseudoscalar meson matrices.

In this paper we adopt the following extended linear sigma model [4]:

$$\mathcal{L}_{\text{light}} = \frac{1}{2}\text{Tr}(\partial_\mu M \partial^\mu M^\dagger) + \frac{1}{2}\text{Tr}(\partial_\mu M' \partial^\mu M'^\dagger)$$
$$- V_0(M, M') - V_{\text{SB}}, \qquad (4)$$

where the first two terms are the kinetic terms for the $q\bar{q}$ and $qq\bar{q}\bar{q}$ states, $V_0(M, M')$ is the potential term invariant under the chiral $SU(3)_L \times SU(3)_R$ transformation, and V_{SB} stands for the explicit chiral symmetry breaking terms due to the current quark masses. It should be noted that $V_0(M, M')$ is decomposed as

$$V_0(M, M') = V_{\text{inv}}(M, M') + V_\eta(M, M'), \qquad (5)$$

where V_{inv} is invariant under the $U(1)_A$ transformation while V_η violates the $U(1)_A$ symmetry explicitly due to the anomaly. We do not specify the form of V_0 in our present analysis but assume that this model allows for spontaneous chiral symmetry breaking by a consistent choice of the parameters in V_0. Since the $U(1)_A$ symmetry is explicitly broken by the anomaly, the form of the $U(1)_A$ violating term V_η is constrained by the anomaly matching condition.

The $U(1)_A$ symmetry breaking by anomaly causes a mixing between the $q\bar{q}$ states and the $qq\bar{q}\bar{q}$ states, as pointed in Refs. [5–7]. In addition, the spontaneous chiral symmetry breaking also generates a mixing between the $q\bar{q}$ states and $qq\bar{q}\bar{q}$ states [5, 7]. As a result, four physical iso-singlet scalar mesons are given as the mixing states of two $q\bar{q}$ states and two $qq\bar{q}\bar{q}$ states through the mixing matrix U_f as

$$\begin{pmatrix} f_{p1} \\ f_{p2} \\ f_{p3} \\ f_{p4} \end{pmatrix} = \begin{pmatrix} (U_f)_{1a} & (U_f)_{1b} & (U_f)_{1c} & (U_f)_{1d} \\ (U_f)_{2a} & (U_f)_{2b} & (U_f)_{2c} & (U_f)_{2d} \\ (U_f)_{3a} & (U_f)_{3b} & (U_f)_{3c} & (U_f)_{3d} \\ (U_f)_{4a} & (U_f)_{4b} & (U_f)_{4c} & (U_f)_{4d} \end{pmatrix} \begin{pmatrix} f_a \\ f_b \\ f_c \\ f_d \end{pmatrix}, \qquad (6)$$

where $f_{p1}, \ldots, f_{p4}$ are the physical scalar mesons with mass ordering $m_{f_1} \leq m_{f_2} \leq m_{f_3} \leq m_{f_4}$. In the following, consistently with PDG notation, we use the notation σ for the lightest scalar meson f_{p1}. f_a and f_b are the $q\bar{q}$ states with $f_a = (S_1^1 + S_2^2)/\sqrt{2}$ and $f_b = S_3^3$ and f_c and f_d are the $qq\bar{q}\bar{q}$ states with $f_c = ((S')_1^1 + (S')_2^2)/\sqrt{2}$ and $f_b = (S')_3^3$.

Similarly, for the two iso-triplet pseudoscalar mesons, we have

$$\begin{pmatrix} \pi_p \\ \pi'_p \end{pmatrix} = \begin{pmatrix} \cos\theta_\pi & -\sin\theta_\pi \\ \sin\theta_\pi & \cos\theta_\pi \end{pmatrix} \begin{pmatrix} \pi \\ \pi' \end{pmatrix}, \qquad (7)$$

where π_p and π'_p in the left-hand side are the physical states while π and π' in the right-hand side denote the $q\bar{q}$ states and $qq\bar{q}\bar{q}$ states, respectively. In the present analysis we identify π_p as $\pi(140)$. As was shown in Refs. [5, 10], the above pseudoscalar mixing angle θ_π relates to the pion decay constant and vacuum expectation

values (VEVs) of the $q\bar{q}$ and $qq\bar{q}\bar{q}$ scalar fields. In the chiral limit, we have

$$F_\pi \cos\theta_\pi = 2v_2, \qquad F_\pi \sin\theta_\pi = -2v_4, \qquad (8)$$

where $F_\pi = 130.41$ MeV [2] stands for the decay constant of $\pi(140)$ and v_2 and v_4 denote the VEVs of the $q\bar{q}$ and the $qq\bar{q}\bar{q}$ scalar fields, respectively.

Using the method shown in Ref. [5], we obtain the relation among the scalar meson-π-π coupling constant $g_{f_j\pi\pi}$, the mixing matrices and the scalar mass as: (for a derivation, see Appendix A)

$$g_{f_j\pi\pi} = \frac{\sqrt{2}}{F_\pi}[\cos\theta_\pi(U_f)_{ja} - \sin\theta_\pi(U_f)_{jc}]m_{f_j}^2, \quad (9)$$

which is similar to the σ-π-π coupling of Ref. [4] except the mixing angle included and the chiral limit taken. We also obtain the relation for the four-pion coupling constant $g_{\pi\pi\pi\pi}$ as (see Appendix A)

$$g_{\pi\pi\pi\pi} = \frac{6}{F_\pi^2}\sum_{j=1}^{4}[\cos\theta_\pi(U_f)_{ja} - \sin\theta_\pi(U_f)_{jc}]^2 m_{f_j}^2. \qquad (10)$$

Making use of the relations (9) and (10) together with the orthonormal conditions,

$$\sum_{j=1}^{4}(U_{ja})^2 = \sum_{j=1}^{4}(U_{jc})^2 = 1,$$
$$\sum_{j=1}^{4}U_{ja}U_{jc} = 0, \qquad (11)$$

we obtain the following sum rules:

$$\sum_{j=1}^{4}\frac{g_{f_j\pi\pi}^2}{m_{f_j}^2} = \frac{1}{3}g_{\pi\pi\pi\pi}, \qquad (12)$$

$$\sum_{j=1}^{4}\frac{g_{f_j\pi\pi}^2}{(m_{f_j}^2)^2} = \frac{2}{F_\pi^2}. \qquad (13)$$

We should note that, as will be shown in the next section, these sum rules guarantee that the π-π scattering amplitude satisfies the low-energy theorem of the Nambu-Goldstone (NG) bosons.

III. SIGMA MESON IN THE π-π SCATTERING

In this section, we determine the sigma meson mass m_σ and σ-π-π coupling constant $g_{\sigma\pi\pi}$ using the isospin zero ($I = 0$), S-wave π-π scattering data.

The matrix element for the isospin zero π-π scattering is expressed as

$$A^{I=0} = 3A(s, t, u) + A(t, u, s) + A(u, s, t), \qquad (14)$$

where s, t and u are the Mandelstam variables and the amplitude $A(s, t, u)$ is defined in such a way that the invariant amplitude for $\pi_i(p_1) + \pi_j(p_2) \to \pi_k(p_3) + \pi_l(p_4)$ is decomposed as

$$\delta_{ij}\delta_{kl}A(s,t,u) + \delta_{ik}\delta_{jl}A(t,s,u) + \delta_{il}\delta_{jk}A(u,t,s) . \quad (15)$$

The partial wave amplitude is obtained as

$$T_L^{I=0} = \frac{1}{2}\rho(s) \int_{-1}^{+1} d\cos\theta P_L(\cos\theta) A^{I=0}(s,t,u), (16)$$

where $\rho(s) = |\mathbf{q}(s)|/(16\pi\sqrt{s})$ with $|\mathbf{q}(s)| = \sqrt{s - 4m_\pi^2}/2$ and $P_L(\cos\theta)$ is the Legendre function.

In the present model, the scattering matrix $A(s, t, u)$ is expressed as

$$A(s,t,u) = -\frac{1}{3}g_{\pi\pi\pi\pi} - \sum_{j=1}^{4} g_{f_j\pi\pi}^2 \frac{1}{s - m_{f_j}^2} . \quad (17)$$

It should be noticed that the sum rule (13) implies that

$$A(s,t,u) = \frac{2s}{F_\pi^2} , \quad \text{for } s \ll m_\sigma^2 , \quad (18)$$

consistently with the low-energy theorem of the NG bosons. After the partial wave and the isospin projection, the matrix element for the iso-singlet S-wave channel is obtained as

$$T_{L=0}^{I=0}(s) = \rho(s)\left[-\frac{5}{3}g_{\pi\pi\pi\pi} + 2\frac{1}{s-4m_\pi^2}\sum_{j=1}^{4} g_{f_j\pi\pi}^2 \ln\left(\frac{s+m_{f_j}^2-4m_\pi^2}{m_{f_j}^2}\right) - 3\sum_{j=1}^{4} g_{f_j\pi\pi}^2 \frac{1}{s-m_{f_j}^2} \right] , \quad (19)$$

where the second term comes from the t- and u-channel scalar exchange contributions, and the third term comes from the s-channel exchange. In this expression, m_π stands for the $\pi(140)$ mass.

Since the rescattering effect should be properly included in the energy region above 800 MeV [13], the above amplitude is applicable only in the low energy region. Actually, in the calculation of the $D_1 \to D\pi\pi$ decay in section V, the exchanged energy is $\sqrt{s} < 560$ MeV, so that, among the exchanged four scalar mesons, the sigma meson gives a dominant contribution. One might naively

eliminate the contributions of f_{p2}, f_{p3} and f_{p4} from the second and third terms in Eq. (17) to reduce the number of parameters. However, such a truncated amplitude cannot reproduce the low-energy theorem in Eq. (18) obtained as a consequence of the chiral symmetry. Then, instead of naively eliminating the scalar mesons other than sigma, we make an expansion of the amplitude in terms of $s/m_{f_j}^2$ and $m_\pi^2/m_{f_j}^2$ $(j = 2, 3, 4)$. As a result, the partial wave amplitude $T_{L=0}^{I=0}(s)$ is reduced to

$$T_{L=0}^{I=0}(s) = \rho(s)\left[-\frac{5}{3}g_{\pi\pi\pi\pi} + 2\frac{1}{s-4m_\pi^2}g_{\sigma\pi\pi}^2 \ln\left(\frac{s+m_\sigma^2-4m_\pi^2}{m_\sigma^2}\right) - 3g_{\sigma\pi\pi}^2 \frac{1}{s-m_\sigma^2} \right.$$
$$\left. + \sum_{j=2}^{4}\left(5\frac{g_{f_j\pi\pi}^2}{m_{f_j}^2} - \frac{g_{f_j\pi\pi}^2}{(m_{f_j}^2)^2}(s-4m_\pi^2) + 3\frac{g_{f_j\pi\pi}^2}{(m_{f_j}^2)^2}s + \cdots \right) \right] , \quad (20)$$

where dots stand for the higher order terms.

Making use of Eqs. (12) and (13), we arrive at

$$T_{L=0}^{I=0}(s) = \rho(s)\left[-5\frac{g_{\sigma\pi\pi}^2}{m_\sigma^2} + 2\frac{1}{s-4m_\pi^2}g_{\sigma\pi\pi}^2 \ln\left(\frac{s+m_\sigma^2-4m_\pi^2}{m_\sigma^2}\right) - 3g_{\sigma\pi\pi}^2 \frac{1}{s-m_\sigma^2} + \left(\frac{2}{F_\pi^2} - \frac{g_{\sigma\pi\pi}^2}{(m_\sigma^2)^2}\right)(2s+4m_\pi^2) \right] . \quad (21)$$

We would like to stress that the amplitude (21) depends on only two undetermined parameters m_σ and $g_{\sigma\pi\pi}$, while the one in Eq. (19) includes nine ($m_{f_i}, g_{f_i\pi\pi}$ and

$g_{\pi\pi\pi\pi}$). This low energy reduction is achieved by using the relations in Eqs. (12) and (13) which are the consequences of the chiral symmetry. Thus, for studying the

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$D_1 \to D\pi\pi$ decay rate, it is enough to determine the parameters m_σ and $g_{\sigma\pi\pi}$.

To study the π-π scattering we include the finite width effect in the sigma meson propagator. Here, we use the modified Breit-Wigner prescription where the width effect in the sigma meson propagator is taken to be momentum dependent, that is, we make the substitution

$$\frac{1}{s - m_\sigma^2} \to \frac{1}{s - m_\sigma^2 + im_\sigma\Gamma_\sigma(s)}. \tag{22}$$

The sigma meson width has several expressions in literature. In the present analysis, we take the width as the imaginary part of the π-π loop contribution to the σ meson self-energy in the linear sigma model

$$\Gamma(s) = \frac{3g_{\sigma\pi\pi}^2}{32\pi m_\sigma}\sqrt{1 - \frac{4m_\pi^2}{s}} \ . \tag{23}$$

Now, we fit the sigma meson mass m_σ and the coupling constant $g_{\sigma\pi\pi}$ from the π-π scattering data given in Refs. [11, 12] in the low energy region below 560 MeV. The best fitted values are obtained as [1]

$$m_\sigma = 606 \pm 9 \,\text{MeV}, \quad |g_{\sigma\pi\pi}| = 2.16 \pm 0.07 \ \text{GeV},$$

$$\frac{\chi^2}{\text{d.o.f.}} = \frac{3.48}{12} = 0.29 \ . \tag{24}$$

Note that we can determine the absolute value of $g_{\sigma\pi\pi}$, the sign of which becomes relevant in $D_1 \to D\pi\pi$ decay. Here we obtained the $g_{\sigma\pi\pi}$ in the chiral limit. We expect the correction from the inclusion of the pion mass is on the order of m_π^2/m_σ^2 [4], which is about 5%. We show
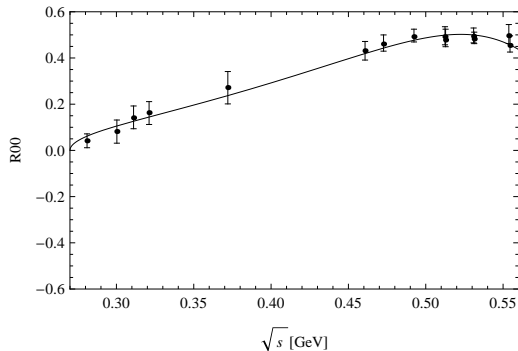

FIG. 1: Best fitted curve of the real part of the $I = 0$ S-wave π-π scattering amplitude. The Data are taken from Refs. [11, 12].

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[1] When we use the parameterization of the width of the σ propagator as $\Gamma(s) = \frac{3g_{\sigma\pi\pi}^2}{32\pi s}\sqrt{1 - \frac{4m_\pi^2}{s}}$ [4], the best fitted values are $m_\sigma = 586 \pm 7\,\text{MeV}$, $|g_{\sigma\pi\pi}| = 1.90 \pm 0.04$ GeV with $\chi^2/\text{d.o.f.} = 7.05/12 = 0.59$. On the other hand, when we use $\Gamma = \frac{3g_{\sigma\pi\pi}^2}{32\pi m_\sigma^2}\sqrt{1 - \frac{4m_\pi^2}{m_\sigma^2}}$ we obtain $m_\sigma = 597 \pm 8\,\text{MeV}$, $|g_{\sigma\pi\pi}| = 2.05 \pm 0.06$ GeV with $\chi^2/\text{d.o.f.} = 5.23/12 = 0.44$.

the best fit curve in Fig. 1 with the allowed region of m_σ and $g_{\sigma\pi\pi}$ at 1σ level shown in Fig. 2. We conclude that our model can reproduce the π-π scattering data below 560 MeV quite well.
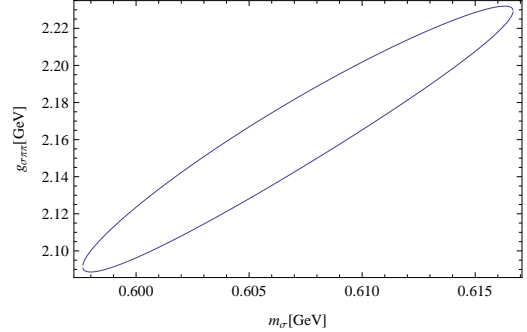

FIG. 2: Allowed region of m_σ and $g_{\sigma\pi\pi}$ at 1σ

IV. EFFECTIVE LAGRANGIAN FOR HEAVY-LIGHT MESONS WITH CHIRAL PARTNER STRUCTURE

In this section, we introduce an effective Lagrangian for the heavy-light mesons coupling to the light mesons based on the heavy quark symmetry combined with the chiral symmetry. For constructing the Lagrangian invariant under the linearly realized chiral symmetry, we need to include the chiral partner to the lowest lying heavy quark multiplet of $H = (D^*, D)$. In the present analysis, we regard the doublet $G = (D_1, D_0^*)$ as the chiral partner to the H doublet [8, 9]. For constructing the effective Lagrangian including these fields, it is essential to consider the symmetry properties, especially the $U(1)_A$ charge, of these heavy-light mesons. Since the mass spectra of the states in the H and G doublets are consistent with the theoretical predictions based on the $c\bar{q}$ interpretation, it is reasonable to regard them as the $c\bar{q}$ states. Following Refs. [8, 9, 14, 15], we include the H and G doublets into the Lagrangian through

$$\mathcal{H}_L = \frac{1}{\sqrt{2}}[G + iH\gamma_5], \quad \mathcal{H}_R = \frac{1}{\sqrt{2}}[G - iH\gamma_5] \ . \tag{25}$$

In terms of the physical states, the H and G doublets are expressed as

$$H = \frac{1 + \not{v}}{2}[D^{*\,\mu}\gamma_\mu + iD\gamma_5],$$

$$G = \frac{1 + \not{v}}{2}[-iD_1^\mu\gamma_\mu\gamma_5 + D_0^*], \tag{26}$$

with v^μ being the velocity of the heavy meson.

Under chiral transformation, the $\mathcal{H}_L$ and $\mathcal{H}_R$ fields transform as

$$\mathcal{H}_L \to \mathcal{H}_L g_L^\dagger, \quad \mathcal{H}_R \to \mathcal{H}_R g_R^\dagger. \tag{27}$$

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under the axial $U(1)_A$ symmetry, $\mathcal{H}_L$ and $\mathcal{H}_R$ transform as

$$\mathcal{H}_L \to \mathcal{H}_L e^{-i\alpha}, \quad \mathcal{H}_R \to \mathcal{H}_R e^{i\alpha}. \quad (28)$$

Using the chiral and $U(1)_A$ transformation properties of heavy-light meson fields and light meson fields, we construct an effective Lagrangian describing the interactions among the heavy-light mesons and the light mesons. In the present construction, we only include the minimal number of terms which are responsible for our following analysis of the $D_1 \to D\pi\pi$ decay. Then the Lagrangian is written as

$$
\begin{aligned}
\mathcal{L}_{\text{heavy}} &= \frac{1}{2}\text{Tr}\left[\bar{\mathcal{H}}_L i(v\cdot\partial)\mathcal{H}_L\right] + \frac{1}{2}\text{Tr}\left[\bar{\mathcal{H}}_R i(v\cdot\partial)\mathcal{H}_R\right] - \frac{\Delta}{2}\text{Tr}\left[\bar{\mathcal{H}}_L\mathcal{H}_L + \bar{\mathcal{H}}_R\mathcal{H}_R\right] \\
&\quad - \frac{g_\pi}{4}\text{Tr}\left[M^\dagger\bar{\mathcal{H}}_L\mathcal{H}_R + M\bar{\mathcal{H}}_R\mathcal{H}_L\right] + i\frac{g_A}{2F_\pi}\text{Tr}\left[\gamma^5\slashed{\partial}M^\dagger\bar{\mathcal{H}}_L\mathcal{H}_R - \gamma^5\slashed{\partial}M\bar{\mathcal{H}}_R\mathcal{H}_L\right],
\end{aligned} \quad (29)
$$

where Δ, g_π and g_A are parameters. In terms of the H and G doublets, the effective Lagrangian (29) is rewritten as

$$
\begin{aligned}
\mathcal{L}_{\text{heavy}} &= \frac{1}{2}\text{Tr}\left[-\bar{H}iv\cdot\partial H + \bar{G}iv\cdot\partial G\right] - \frac{\Delta}{2}\text{Tr}\left[-\bar{H}H + \bar{G}G\right] \\
&\quad - \frac{g_\pi}{8}\text{Tr}\left[(M + M^\dagger)(\bar{G}G + \bar{H}H) - i(M - M^\dagger)(\bar{H}G - \bar{G}H)\gamma^5\right] \\
&\quad + i\frac{g_A}{4F_\pi}\text{Tr}\left[-(\slashed{\partial}M - \slashed{\partial}M^\dagger)(\bar{G}G + \bar{H}H)\gamma^5 - i(\slashed{\partial}M + \slashed{\partial}M^\dagger)(\bar{H}G + \bar{G}H)\right].
\end{aligned} \quad (30)
$$

V. THE SIGMA MESON STRUCTURE FROM $D_1 \to D\pi\pi$ DECAY

In this section, we study the quark contents of the σ meson through the $D_1 \to D\pi\pi$ decay.

A. Determination of the parameters in the heavy meson effective Lagrangian

In this subsection, we determine the parameters in the Lagrangian (30) that are necessary for the numerical calculation of the $D_1 \to D\pi\pi$ decay width.

First, since the kinetic terms of H and G doublets have the opposite signs, the Δ term in Eq. (30) shifts the masses of H doublet and G doublet to the same direction. Then, in the following analysis, we take $\Delta = 0$ without loss of generality.

Second, we fix the value of the combination $g_A\cos\theta_\pi$ from the partial width for $D^* \to D\pi$ decay. From the Lagrangian (30), the D^*-D-π interaction is extracted as

$$\mathcal{L}_{D^*D\pi} = -\frac{ig_A}{F_\pi}D\partial_\mu\Phi D^{*\mu\dagger} + \text{H.c.}. \quad (31)$$

From this the $D^* \to D\pi$ is expressed as

$$\Gamma(D^{*+} \to D\pi) = \frac{(g_A\cos\theta_\pi)^2}{4\pi F_\pi^2}\frac{m_H^2}{m_{D^*}^2}|\mathbf{p}_\pi|^3, \quad (32)$$

where $|\mathbf{p}_\pi|$ is three-momentum of outgoing pion and m_H is the average mass defined as $m_H = (3m_{D^*} + m_D)/4$. Using the central value for $\Gamma_{D^{*+}} = 96$ KeV [2] we get

$$|g_A\cos\theta_\pi| = 0.56. \quad (33)$$

Third, we determine the combination $g_\pi\cos\theta_\pi$ using the $D_0^* \to D\pi$ decay. The relevant terms in the Lagrangian (30) are given by

$$\mathcal{L}_{D_0^*D\pi} = \frac{ig_\pi}{2}D\Phi D_0^{*\dagger} + \text{H.c.}, \quad (34)$$

From this Lagrangian, the width for $D_0^* \to D\pi$ decay is expressed as

$$\Gamma(D_0^* \to D\pi) = \frac{3}{2}\frac{m_H m_G}{2\pi m_{D_0^*}^2}\frac{(g_\pi\cos\theta_\pi)^2}{4}|\mathbf{p}_\pi|, \quad (35)$$

where m_G is the average mass defined as $m_G = (3m_{D_1} + m_{D_0^*})/4$. Using the central value of the data $\Gamma(D_0^*) = 267$ MeV [2] and assuming that D_0^* decays dominantly to $D\pi$, we obtain

$$|g_\pi\cos\theta_\pi| = 3.61. \quad (36)$$

With the numerical value of $|g_\pi\cos\theta_\pi|$ estimated above we predict the $D_1 \to D^*\pi$ decay width as a check of the validity of the heavy quark expansion applied here. From the Lagrangian (30) we obtain the relevant Lagrangian as

$$\mathcal{L}_{D_1 D^*\pi} = -\frac{ig_\pi}{2}D_\mu^*\Phi D_1^{\mu\dagger} + \text{H.c.}, \quad (37)$$

which yields the following expression of the $D_1 \to D^*\pi$ decay width:

$$\Gamma(D_1 \to D^*\pi) = \frac{3}{2}\frac{m_H m_G}{2\pi m_{D_1}^2}\frac{(g_\pi\cos\theta_\pi)^2}{4}|\mathbf{p}_\pi|. \quad (38)$$

Using the central values for the relevant particle masses we obtain

$$\Gamma(D_1 \to D^*\pi) = 224.0 \text{ MeV}. \qquad (39)$$

Comparing this with the data $\Gamma(D_1) = 383^{+107}_{-75} \pm 74$ [2] we conclude that our prediction, based on the heavy quark limit, is consistent with the data. This implies that the heavy quark limit taken here is a reasonable

approach for the H and G doublets.

B. $D_1 \to D\pi\pi$ decay

In this subsection, using the model explained so far, we study the $D_1 \to D\pi\pi$ decay. We show the relevant diagrams in the present model in Fig. 3.
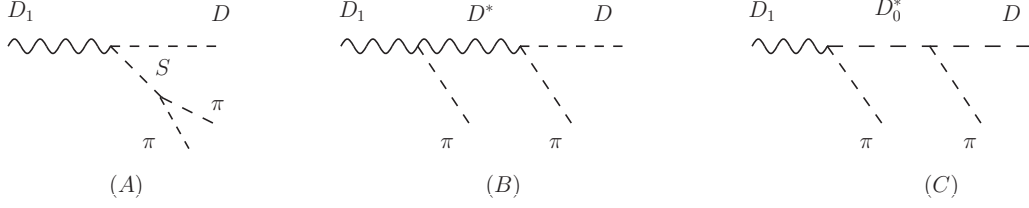
FIG. 3: Feynman diagrams contributing to $D_1 \to D\pi\pi$ decay in our model.

The matrix element for this process is straightforwardly written as

$$\mathcal{M} = -\sqrt{2m_G m_H}\frac{g_A}{F_\pi}\epsilon_\mu(v)\left\{\sum_{i=1}^{4} g_{f_i\pi\pi}\frac{(p_{\pi_1}+p_{\pi_2})^\mu}{s - m_{f_i}^2 + im_{f_i}\Gamma_{f_i}(s)}(U_f^{-1})_{ai}\right.$$
$$\left. + \frac{g_\pi}{2\sqrt{2}}\cos^2\theta_\pi\frac{p_{\pi_2}^\mu - v^\mu v\cdot p_{\pi_2}}{v\cdot k_{D^*} + i\Gamma_{D^*}/2} + \frac{g_\pi}{2\sqrt{2}}\cos^2\theta_\pi\frac{p_{\pi_1}^\mu}{v\cdot k_{D_0^*} + i\Gamma_{D_0^*}/2}\right\}, \qquad (40)$$

where $s = (p_{\pi_1} + p_{\pi_2})^2 = m_{\pi\pi}^2$ with p_{π_1} and p_{π_2} as the momenta of pions in the final states, $\epsilon_\mu(v)$ is the polarization vector of D_1 meson satisfying $\sum_{\text{polarization}} \epsilon_\mu \epsilon_\nu^* = -(g_{\mu\nu} - v_\mu v_\nu)$, and the residual momenta k_{D^*} and $k_{D_0^*}$ are defined by $p_{D^*}^\mu = m_{D^*}v^\mu + k_{D^*}^\mu$ and $p_{D_0^*}^\mu = m_{D_0^*}v^\mu + k_{D_0^*}^\mu$, respectively.

We note that, since the maximum value of the $\pi\pi$ invariant mass is restricted as $m_{\pi\pi} \leq m_{D_1} - m_D \simeq 560$ MeV, the contribution from the lightest scalar meson, the sigma meson is dominant. Along the same method as that was used in the π-π scattering case, using the sum rules (12) and (13), the matrix element (40) is reduced as (for a derivation, see Appendix B)

$$\mathcal{M} = -\sqrt{2m_G m_H}\frac{g_A\cos\theta_\pi}{F_\pi}\epsilon_\mu(v)\left\{h\,g_{\sigma\pi\pi}\frac{(p_{\pi_1}+p_{\pi_2})^\mu}{s - m_\sigma^2 + im_\sigma\Gamma_\sigma(s)} - (p_{\pi_1}+p_{\pi_2})^\mu\left(\frac{\sqrt{2}}{F_\pi} - \frac{h\,g_{\sigma\pi\pi}}{m_\sigma^2}\right)\right.$$
$$\left. + \frac{g_\pi\cos\theta_\pi}{2\sqrt{2}}\frac{p_{\pi_2}^\mu - v^\mu v\cdot p_{\pi_2}}{v\cdot k_{D^*} + i\Gamma_{D^*}/2} + \frac{g_\pi\cos\theta_\pi}{2\sqrt{2}}\frac{p_{\pi_1}^\mu}{v\cdot k_{D_0^*} + i\Gamma_{D_0^*}/2}\right\}, \qquad (41)$$

where the mixing parameter h is defined as

$$h = \frac{(U_f^{-1})_{a1}}{\cos\theta_\pi}. \qquad (42)$$

In the amplitude (41), the values of $|g_\pi\cos\theta_\pi|$ and $|g_A\cos\theta_\pi|$ as well as those of m_σ and $|g_{\sigma\pi\pi}|$ are phenomenologically determined above. Then the amplitude (41) depends only on the mixing parameter h. When the physical pion is a pure $q\bar{q}$ state, i.e., $\cos\theta_\pi = 1$, then $h = 1$ implies that σ meson is the $q\bar{q}$ state while $h = 0$ the pure $qq\bar{q}\bar{q}$ state.

Next, we make the isospin and the partial wave projection with respect to the final two pions and pick up the $I = 0, S$-wave amplitude. This procedure eliminates the contributions from $L = 2, 4, \cdots$ in diagrams (B) and (C) in

Fig. 3. [2] The isospin projection gives the isospin singlet as

$$|\pi\pi, I = 0\rangle = \frac{1}{\sqrt{3}} \left| \pi^+(p_1)\pi^-(p_2) + \pi^-(p_1)\pi^+(p_2) + \pi^0(p_1)\pi^0(p_2) \right\rangle. \tag{43}$$

And, the partial wave projection decompose the $D_1 \to D\pi\pi$ decay matrix as

$$\frac{d\Gamma(D_1 \to D\pi\pi)}{dm_{\pi\pi}} = \sum_L \frac{d\Gamma_L(D_1 \to D\pi\pi)}{dm_{\pi\pi}}, \tag{44}$$

where

$$\frac{d\Gamma_L(D_1 \to D\pi\pi)}{dm_{\pi\pi}} = 2(2L+1)\frac{1}{2!}\frac{1}{64\pi^3 m_{D_1}}\left(\frac{1}{2}\right)\frac{|\mathbf{p}_\pi|^2}{E_D}\sum_{\text{polarization}}\left|\mathcal{M}_L(m_{\pi\pi}^2)\right|^2, \tag{45}$$

with $|\mathbf{p}_\pi| = \sqrt{m_{\pi\pi}^2 - 4m_\pi^2}/2$ as the three momentum of the outgoing pions in the center of mass frame of two pions. $\mathcal{M}_L(m_{\pi\pi}^2)$ stands for the $D_1 \to D\pi\pi$ decay matrix element with the final two pions in L-wave, explicitly,

$$\mathcal{M}_L(m_{\pi\pi}^2) = \frac{1}{2}\int_{-1}^{1} d\cos\theta P_L(\cos\theta)\mathcal{M}(m_{\pi\pi}, \cos\theta), \tag{46}$$

with $\mathcal{M}$ given by Eq. (41) and θ being the angle between π and D. In the following, we only consider the $I = 0, S$-wave $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$ to study the composition of the σ meson. Taking into account of the sign ambiguities of $g_{\sigma\pi\pi}$ and $g_\pi \cos\theta_\pi$, we show, in Fig. 4, the resultant width with $h = 0$ (solid line) and $h = 1$ (dotted line) for four cases: (a) $g_{\sigma\pi\pi} > 0$ and $g_\pi \cos\theta_\pi > 0$; (b) $g_{\sigma\pi\pi} > 0$ and $g_\pi \cos\theta_\pi < 0$; (c) $g_{\sigma\pi\pi} < 0$ and $g_\pi \cos\theta_\pi > 0$; and (d) $g_{\sigma\pi\pi} < 0$ and $g_\pi \cos\theta_\pi < 0$. (Since $g_A \cos\theta_\pi$ is the overall factor of the amplitude, the sign ambiguity is irrelevant to the decay width.) This shows that, for $g_{\sigma\pi\pi} > 0$, the width for $h = 1$ is much larger than that for $h = 0$. On the other hand, for $g_{\sigma\pi\pi} < 0$, the peak position is located around 450 MeV for $h = 0$ while above 500 MeV for $h = 1$. This suggests that, when the experimental data become available in the future, the mixing parameter h together with the signs of $g_{\sigma\pi\pi}$ and $g_{\sigma\pi\pi}$ could be fitted.

VI.   A SUMMARY AND DISCUSSIONS

We study how the composition of the lightest scalar meson σ affects the $D_1 \to D\pi\pi$ decay. We construct a linear sigma model for the $q\bar{q}$ and $qq\bar{q}\bar{q}$ chiral nonet fields, distinguishing two nonets by their $U(1)_A$ charge. We write down the effective interaction terms among $c\bar{q}$-type heavy mesons ($D, D^*, D_0^*(2400), D_1(2430)$) and the scalar nonets, in which the existence of the $U(1)_A$ symmetry implies that only the $q\bar{q}$ component of the scalar meson couples to the heavy mesons.

In the light quark sector, we use the most general form of the potential for the two chiral nonet fields. Using the sum rules among the coupling constants and masses of scalar mesons, we express the $\pi\pi$ scattering amplitude in the low-energy region in terms of only two parameters, m_σ and $g_{\sigma\pi\pi}$. We fit the values of m_σ and

$g_{\sigma\pi\pi}$ to the $\pi\pi$ scattering data below 560 MeV, which is the maximum energy transferred to two pions in the $D_1 \to D\pi\pi$ decay. We fix the parameters in the effective interaction terms to heavy meson phenomenologically. We show how the $I = 0, S$-wave differential decay rate, $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$, depends on the mixing structure of the σ meson: For $g_{\sigma\pi\pi} > 0$, the width for $h = 1$ is much larger than that for $h = 0$, and the peak position moves to the higher energy region for $g_{\sigma\pi\pi} < 0$.

In the theoretical calculation, we can include a part of the final state interaction effect. The final state interaction among all the three final particles and that between one pion and D meson are suppressed by the heavy meson mass therefore their effects are small. For the final state interaction between the two pions, in the $I = 0, S$-wave channel, the contribution from the bubble diagrams is illustrated as
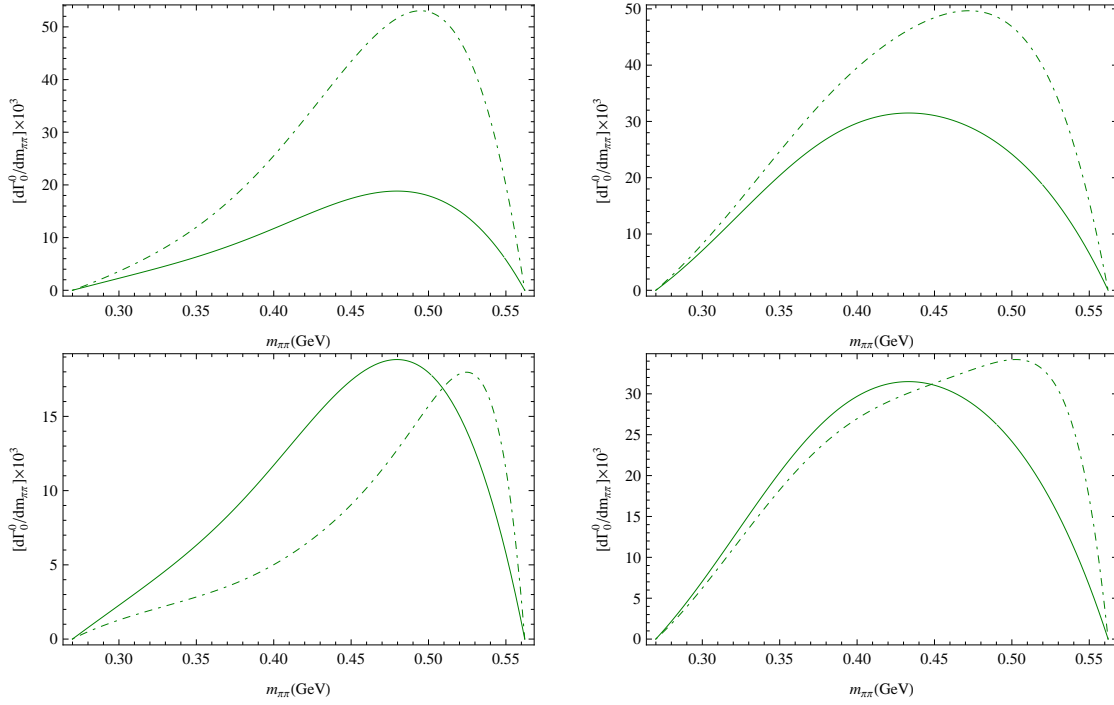
FIG. 4: $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$ vs $m_{\pi\pi}$ with $h = 0$ (solid line) and $h = 1$ (dotted line) for $g_{\sigma\pi\pi} > 0$ and $g_\pi \cos\theta_\pi > 0$ (upper left panel), $g_{\sigma\pi\pi} > 0$ and $g_\pi \cos\theta_\pi < 0$ (upper right panel), $g_{\sigma\pi\pi} < 0$ and $g_\pi \cos\theta_\pi > 0$ (lower left panel), $g_{\sigma\pi\pi} < 0$ and $g_\pi \cos\theta_\pi < 0$ (lower right panel).

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[2] We calculated the D-wave decay width which is less than 5% of the S-wave width.

The summation of the terms in the second parenthesis in the right hand side of the above equation gives the $I = 0, S$-wave S-matrix for the π-π scattering, therefore it just contributes a phase factor to the $D_1 \to D\pi\pi$ decay matrix element and further, does not change the partial width. In this sense, the correction to our results from the final state interaction is small.

In the present analysis, the interaction terms among the heavy-light mesons and light mesons is constructed in the heavy quark limit with the linear realization of

the chiral symmetry, therefore the $D_1 D\sigma$ and $D^* D\pi$ coupling constants are related by the chiral symmetry. Since the chiral symmetry is dynamically broken, there exists a difference between the $D_1 D\sigma$ and $D^* D\pi$ coupling constants. To show this chiral symmetry breaking effect, we typically rescale the $D_1 D\sigma$ coupling constant by a factor two and illustrate out results in Fig. 5. This shows that the tendency of the differential width $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$ does not change.

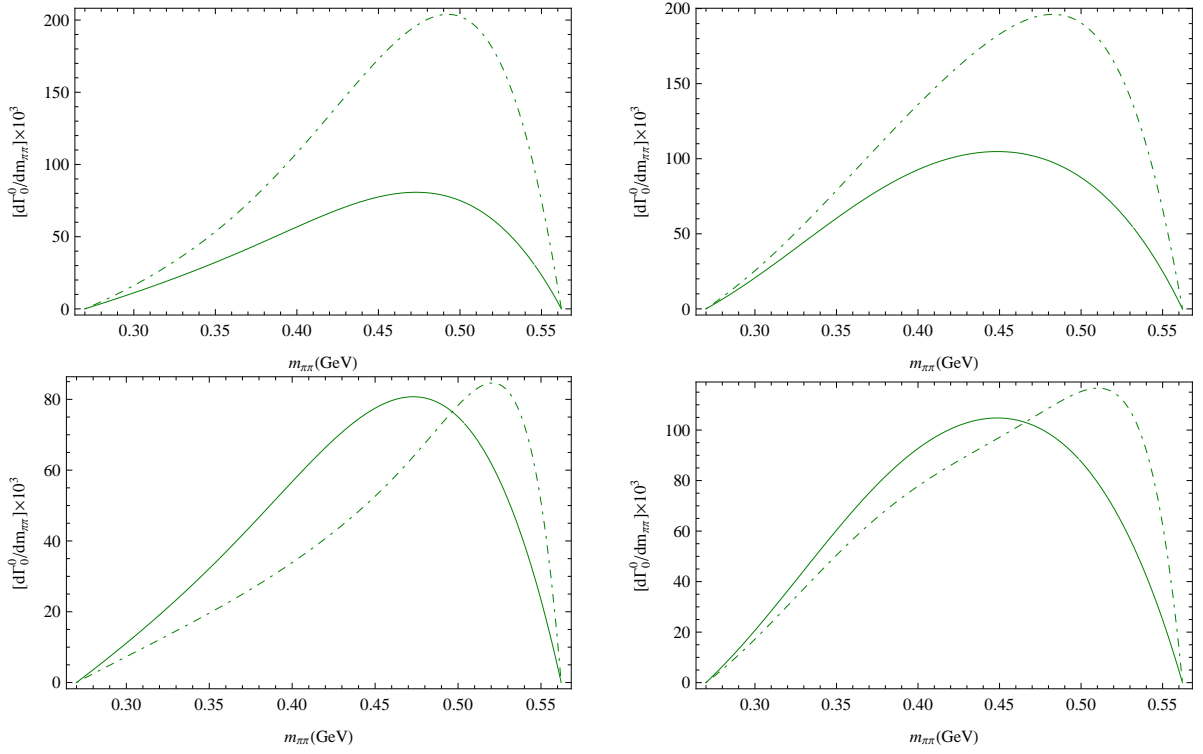FIG. 5: $d\Gamma(D_1 \to D(\pi\pi)_{I=0,L=0})/dm_{\pi\pi}$ vs $m_{\pi\pi}$ with $h=0$ (solid line) and $h=1$ (dotted line) for $g_{\sigma\pi\pi} > 0$ and $g_\pi \cos\theta_\pi > 0$ (upper left panel), $g_{\sigma\pi\pi} > 0$ and $g_\pi \cos\theta_\pi < 0$ (upper right panel), $g_{\sigma\pi\pi} < 0$ and $g_\pi \cos\theta_\pi > 0$ (lower left panel), $g_{\sigma\pi\pi} < 0$ and $g_\pi \cos\theta_\pi < 0$ (lower right panel).

Acknowledgments

Appendix A: Derivation of the coupling constants

In this appendix, we derive the relations (9) and (13) from the general Lagrangian (4). Following Ref. [10], we write the infinitesimal transformation matrices for vector and axial-vector transformations as E_V and E_A, respectively. These infinitesimal matrices satisfy the following relations:

$$E_V^\dagger = -E_V, \qquad E_A^\dagger = -E_A, \qquad \mathrm{Tr}E_A = 3i\alpha.$$

Then, under the vector and axial-vector transformations, the relevant meson fields transform as

$$
\begin{aligned}
\delta_V \Phi &= [E_V, \Phi], & \delta_V S &= [E_V, S], \\
\delta_A \Phi &= -i\{E_A, S\}, & \delta_A S &= i\{E_A, \Phi\}, \\
\delta_V \Phi' &= [E_V, \Phi'], & \delta_V S' &= [E_V, S'], \\
\delta_A \Phi' &= -i\{E_A, S'\} + 2iS'\mathrm{Tr}E_A, \\
\delta_A S' &= i\{E_A, \Phi'\} - 2i\Phi'\mathrm{Tr}E_A.
\end{aligned}
\tag{A1}
$$

Since the effective potential term V_0 is invariant under the chiral $SU(3)_L \times SU(3)_R$ transformation implying the invariance under the vector transformation, then we have

$$\delta_V V_0 = \mathrm{Tr}\left[\frac{\partial V_0}{\partial \Phi}\delta_V \Phi + \frac{\partial V_0}{\partial S}\delta_V S\right] + [(S, \Phi) \to (S', \Phi')] = 0. \tag{A2}$$

The $U(1)_A$ symmetry is explicitly broken by the anomaly in V_η, so that the axial transformation is expressed as

$$\delta_A V_0 = \mathrm{Tr}\left[\frac{\partial V_0}{\partial \Phi}\delta_A \Phi + \frac{\partial V_0}{\partial S}\delta_A S\right] + [(S, \Phi) \to (S', \Phi')] = \delta V_\eta, \tag{A3}$$

where δV_η arises from the chiral anomaly. Then, using Eqs.(A1), (A2) and (A3) as well as the arbitrariness of the variations E_V and E_A yields the following generating equations:

$$
\begin{aligned}
&\left[\Phi, \frac{\partial V_0}{\partial \Phi}\right] + \left[S, \frac{\partial V_0}{\partial S}\right] + (S, \Phi) \to (S', \Phi') = 0, \\
&\left\{\Phi, \frac{\partial V_0}{\partial S}\right\} - \left\{S, \frac{\partial V_0}{\partial \Phi}\right\} + (S, \Phi) \to (S', \Phi') = \mathbf{1}\left[2\mathrm{Tr}\left[\Phi'\frac{\partial V_0}{\partial S'} - S'\frac{\partial V_0}{\partial \Phi'}\right] + \frac{\delta V_\eta}{3i\alpha}\right].
\end{aligned}
\tag{A4}
$$

In the following analysis, we use the stationary conditions for the potential V_0 given by

$$\left\langle\frac{\partial V_0}{\partial S}\right\rangle = 0, \qquad \left\langle\frac{\partial V_0}{\partial S'}\right\rangle = 0.$$

Furthermore, we work in the chiral limit, therefore VEVs of S and S' are proportional to the unit matrix:

$$\langle S_i{}^j\rangle = v_2 \delta_i{}^j, \qquad \langle S_i'{}^j\rangle = v_4 \delta_i{}^j.$$

Differentiating Eq. (A4) with respect to Φ, we obtain

$$2v_2 \left\langle\frac{\partial^2 V_0}{\partial \Phi_i^j \partial \Phi_k^l}\right\rangle + 2v_4 \left\langle\frac{\partial^2 V_0}{\partial \Phi_i^j \partial \Phi_k'^l}\right\rangle = \delta_l^k\left[2v_4 \sum_{m=1}^{3}\left\langle\frac{\partial^2 V_0}{\partial \Phi_i^j \partial \Phi_m'^m}\right\rangle - \frac{1}{3i\alpha}\left\langle\frac{\partial \delta V_\eta}{\partial \Phi_i^j}\right\rangle\right]. \tag{A5}$$

Differentiation of Eq. (A4) with respect to Φ' gives

$$2v_2 \left\langle\frac{\partial^2 V_0}{\partial \Phi_i'^j \partial \Phi_k^l}\right\rangle + 2v_4 \left\langle\frac{\partial^2 V_0}{\partial \Phi_i'^j \partial \Phi_k'^l}\right\rangle = \delta_l^k\left[2v_4 \sum_{m=1}^{3}\left\langle\frac{\partial V_0}{\partial \Phi_i'^j \partial \Phi_m'^m}\right\rangle - \frac{1}{3i\alpha}\left\langle\frac{\partial \delta V_\eta}{\partial \Phi_i'^j}\right\rangle\right]. \tag{A6}$$

Along the same method, differentiating Eq. (A4) twice with respect to S, S', Φ or Φ' and using

$$
\begin{aligned}
\partial/\partial \pi^0 &= (\partial/\partial \Phi_1^1 - \partial/\partial \Phi_2^2)/\sqrt{2}, \\
\partial/\partial \pi'^0 &= (\partial/\partial \Phi_1'^1 - \partial/\partial \Phi_2'^2)/\sqrt{2},
\end{aligned}
$$

10

$$\partial/\partial f_a = (\partial/\partial S_1^1 + \partial/\partial S_2^2)/\sqrt{2},$$
$$\partial/\partial f_b = \partial/\partial S_3^3,$$
$$\partial/\partial f_c = (\partial/\partial S_1'^1 + \partial/\partial S_2'^2)/\sqrt{2},$$
$$\partial/\partial f_d = \partial/\partial S_3'^3, \tag{A7}$$

we obtain the relations between three-point and two-point couplings in the isospin limit as

$$v_2\left\langle\frac{\partial^3 V_0}{\partial(\pi^0)^2\partial f_a}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_a}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial(f_a)^2}\right\rangle - \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial(\pi^0)^2}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial(\pi^0)^2\partial f_b}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_b}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial f_a\partial f_b}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_a}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial(\pi'^0)^2\partial f_a}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial f_a\partial f_c}\right\rangle - \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial\pi^0\partial\pi'^0}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_b}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial(\pi'^0)^2\partial f_b}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial f_b\partial f_c}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial(\pi^0)^2\partial f_c}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_c}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial f_a\partial f_c}\right\rangle - \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial\pi^0\partial\pi'^0}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial(\pi^0)^2\partial f_d}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_d}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial f_a\partial f_d}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_c}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial(\pi'^0)^2\partial f_c}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial(f_c)^2}\right\rangle - \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial(\pi'^0)^2}\right\rangle,$$

$$v_2\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_d}\right\rangle + v_4\left\langle\frac{\partial^3 V_0}{\partial(\pi'^0)^2\partial f_d}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^2 V_0}{\partial f_c\partial f_d}\right\rangle. \tag{A8}$$

And, differentiating Eq. (A4) three times with respect to S, S', Φ or Φ', the relations between four-point and three-point coupling in the isospin limit are obtained as

$$v_2\left\langle\frac{\partial^4 V_0}{\partial(\pi^0)^4}\right\rangle + v_4\left\langle\frac{\partial^4 V_0}{\partial(\pi^0)^3\partial\pi'^0}\right\rangle = 3\frac{1}{\sqrt{2}}\left\langle\frac{\partial^3 V_0}{\partial(\pi^0)^2\partial f_a}\right\rangle,$$

$$v_2\left\langle\frac{\partial^4 V_0}{\partial\pi'^0\partial(\pi^0)^3}\right\rangle + v_4\left\langle\frac{\partial^4 V_0}{\partial\pi'^0\partial(\pi^0)^2\partial\pi'^0}\right\rangle = 2\frac{1}{\sqrt{2}}\left\langle\frac{\partial^3 V_0}{\partial\pi'^0\partial\pi^0\partial f_a}\right\rangle + \frac{1}{\sqrt{2}}\left\langle\frac{\partial^3 V_0}{\partial(\pi^0)^2\partial f_c}\right\rangle,$$

$$v_2\left\langle\frac{\partial^4 V_0}{\partial\pi^0\partial(\pi'^0)^2\partial\pi^0}\right\rangle + v_4\left\langle\frac{\partial^4 V_0}{\partial\pi^0\partial(\pi'^0)^3}\right\rangle = \frac{1}{\sqrt{2}}\left\langle\frac{\partial^3 V_0}{\partial(\pi'^0)^2\partial f_a}\right\rangle + 2\frac{1}{\sqrt{2}}\left\langle\frac{\partial^3 V_0}{\partial\pi^0\partial\pi'^0\partial f_c}\right\rangle,$$

$$v_2\left\langle\frac{\partial^4 V_0}{\partial(\pi'^0)^3\partial\pi^0}\right\rangle + v_4\left\langle\frac{\partial^4 V_0}{\partial(\pi'^0)^4}\right\rangle = 3\frac{1}{\sqrt{2}}\left\langle\frac{\partial^3 V_0}{\partial(\pi'^0)^2\partial f_c}\right\rangle. \tag{A9}$$

The above relations are re-expressed in terms of the mixing angles for the scalar and pseudoscalar mesons in Eqs. (6) and (7), the pion decay constant and the VEVs of M and M' fields as

$$v_2 = \frac{1}{2}\left[F_\pi\cos\theta_\pi + \tilde{F}_\pi\sin\theta_\pi\right] = \frac{1}{2}\sum_{j=1}^{2}(F_\pi)_j(U_\pi^{-1})_{aj},$$

$$v_4 = \frac{1}{2}\left[-F_\pi\sin\theta_\pi + \tilde{F}_\pi\cos\theta_\pi\right] = \frac{1}{2}\sum_{j=1}^{2}(F_\pi)_j(U_\pi^{-1})_{bj}, \tag{A10}$$

where $(F_\pi)_1 = F_\pi, (F_\pi)_2 = \tilde{F}_\pi$ are the decay constants of $\pi(140)$ and $\pi(1300)$, respectively, and the mixing matrix U_π is defined by

$$U_\pi = \begin{pmatrix} (U_\pi)_{1a} & (U_\pi)_{1b} \\ (U_\pi)_{2a} & (U_\pi)_{2b} \end{pmatrix} = \begin{pmatrix} \cos\theta_\pi & -\sin\theta_\pi \\ \sin\theta_\pi & \cos\theta_\pi \end{pmatrix}.$$

Note that, in the chiral limit, only the NG boson can couple to the axial vector current, so that $\tilde{F}_\pi = 0$. Then, writing the mass matrices as

$$(\tilde{m}_\pi^2)_{AB} \equiv \left\langle \frac{\partial^2 V}{\partial \pi_A \partial \pi_B} \right\rangle, \quad (\tilde{m}_f^2)_{CD} \equiv \left\langle \frac{\partial^2 V}{\partial f_C \partial f_D} \right\rangle,$$

where subscripts A and B run from a to b and C and D from a to d, we have

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_a \partial(\pi^0)_A \partial f_a} \right\rangle = (\tilde{m}_f^2)_{aa} - (\tilde{m}_\pi^2)_{aa},$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_a \partial(\pi^0)_A \partial f_b} \right\rangle = (\tilde{m}_f^2)_{ab},$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_A \partial(\pi^0)_b \partial f_a} \right\rangle = (\tilde{m}_f^2)_{ac} - (\tilde{m}_\pi^2)_{ab}$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_A \partial(\pi^0)_b \partial f_b} \right\rangle = (\tilde{m}_f^2)_{bc},$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_a \partial(\pi^0)_A \partial f_c} \right\rangle = (\tilde{m}_f^2)_{ac} - (\tilde{m}_\pi^2)_{ab},$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_a \partial(\pi^0)_A \partial f_d} \right\rangle = (\tilde{m}_f^2)_{ad},$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_b \partial(\pi^0)_A \partial f_c} \right\rangle = (\tilde{m}_f^2)_{cc} - (\tilde{m}_\pi^2)_{bb},$$

$$\frac{1}{\sqrt{2}} F_\pi \sum_{A=a}^{b} (U_\pi)_{1A} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_b \partial(\pi^0)_A \partial f_d} \right\rangle = (\tilde{m}_f^2)_{cd}, \tag{A11}$$

Summing these equations with mixing matrices U_π and U_f we get the following physical coupling

$$\frac{1}{\sqrt{2}} F_\pi \left\langle \frac{\partial^3 V_0}{\partial(\pi_p^0)_1 \partial(\pi_p^0)_k \partial f_j} \right\rangle = \frac{1}{\sqrt{2}} F_\pi \sum_{A,B=a,b} \sum_{C=a,b,c,d} (U_\pi)_{1A} (U_\pi)_{kB} (U_f)_{jC} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_B \partial(\pi^0)_A \partial f_C} \right\rangle$$

$$= \sum_{C=a}^{d} (U_f)_{jC} \left[(U_\pi)_{ka} (m_f^2)_{aC} + (U_\pi)_{kb} (m_f^2)_{cC} \right]$$

$$- \sum_{A=a}^{b} (U_\pi)_{kA} \left[(U_f)_{ja} (m_\pi^2)_{aA} + (U_f)_{jc} (m_\pi^2)_{Ab} \right], \tag{A12}$$

where we write the physical pseudoscalars as $(\pi_p^0)_1 \equiv \pi_p^0$ and $(\pi_p'^0)_2 \equiv \pi_p'^0$.

So that, in the case that the pseudoscalars are neutral, we have

$$g_{f_j \pi \pi} \equiv \left\langle \frac{\partial^3 V}{\partial(\pi_p^0)_1 \partial(\pi_p^0)_1 \partial f_j} \right\rangle = \sum_{A=a}^{b} \sum_{B=a}^{b} \sum_{C=a}^{d} (U_\pi)_{1A} (U_\pi)_{1B} (U_f)_{jC} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_A \partial(\pi^0)_B \partial f_C} \right\rangle. \tag{A13}$$

As a result

$$g_{f_j \pi \pi}^2 = \left[\sum_{A,B=a,b} \sum_{C=a,b,c,d} (U_\pi)_{1A} (U_\pi)_{1B} (U_f)_{jC} \left\langle \frac{\partial^3 V_0}{\partial(\pi^0)_B \partial(\pi^0)_A \partial f_C} \right\rangle \right]^2$$

$$= \frac{2}{F_\pi^2} \left(m_{f_j}^2 \right)^2 \left[(U_\pi)_{1a} (U_f)_{ja} + (U_\pi)_{1b} (U_f)_{jc} \right]^2. \tag{A14}$$

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Here we have considered that in the chiral limit, the light pions are massless. Therefore, from this equation, we obtain the following relations

$$\sum_{i=1}^{4} \frac{g_{f_j \pi \pi}^2}{\left(m_{f_j}^2\right)^2} = \sum_{i=1}^{4} \frac{2}{F_\pi^2} \left[(U_\pi)_{1a}(U_f)_{ja} + (U_\pi)_{1b}(U_f)_{jc}\right]^2 = \frac{2}{F_\pi^2},$$

$$\sum_{i=1}^{4} \frac{g_{f_j \pi \pi}}{m_{f_j}^2} \left(U_f^{-1}\right)_{aj} = \sum_{i=1}^{4} \frac{\sqrt{2}}{F_\pi} \left[(U_\pi)_{1a}(U_f)_{ja} + (U_\pi)_{1b}(U_f)_{jc}\right] \left(U_f^{-1}\right)_{aj} = \frac{\sqrt{2}}{F_\pi} \cos\theta_\pi. \tag{A15}$$

Similarly, in the the chiral limit, we can derive that 4-π^0 coupling constant satisfies

$$g_{\pi\pi\pi\pi} = \left\langle \frac{\partial^4 V}{\partial(\pi_p^0)_1 \partial(\pi_p^0)_1 \partial(\pi_p^0)_1 \partial(\pi_p^0)_1} \right\rangle = \frac{6}{F_\pi^2} \sum_{j=1}^{4} m_{f_j}^2 \left[(U_\pi)_{1a}(U_f)_{ja} + (U_\pi)_{1b}(U_f)_{jc}\right]^2. \tag{A16}$$

which yields

$$\sum_{i=1}^{4} \frac{g_{f_j \pi \pi}^2}{m_{f_j}^2} = \frac{1}{3} g_{\pi\pi\pi\pi}, \tag{A17}$$

by using Eqs.(A14, A16).

Appendix B: Reduction of the matrix element (40)

Here, we provide the reduction of the heavy meson decay matrix element (40). Its first term can be rewritten as

$$\sum_{i=1}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{s - m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)} (U_f^{-1})_{ai} = g_{f_1 \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{s - m_{f_1}^2 + i m_{f_1} \Gamma_{f_1}(s)} (U_f^{-1})_{a1}$$

$$+ \sum_{i=2}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{s - m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)} (U_f^{-1})_{ai}.$$

Considering that the momentum carried by the exchanged scalar meson is smaller than the heavier scalar meson mass, i.e., $s < m_{f_i}^2, (i = 2, 3, 4)$, we make the expansion

$$\sum_{i=2}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{s - m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)} (U_f^{-1})_{ai} = -\sum_{i=2}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{m_{f_i}^2 - i m_{f_i} \Gamma_{f_i}(s)} (U_f^{-1})_{ai}$$

$$\times \left[1 + \frac{s}{m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)} + \left(\frac{s}{m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)}\right)^2 + \cdots\right].$$

And also concerning that $\Gamma_{f_i} \ll m_{f_i}, (i = 2, 3, 4)$, we can simplify the above relation as

$$\sum_{i=2}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{s - m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)} (U_f^{-1})_{ai} = -\sum_{i=2}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{m_{f_i}^2} (U_f^{-1})_{ai} \left[1 + \frac{s}{m_{f_i}^2} + \left(\frac{s}{m_{f_i}^2}\right)^2 + \cdots\right].$$

With respect to the relation

$$\sum_{i=1}^{4} \frac{g_{f_i \pi \pi}}{m_{f_i}^2} (U_f^{-1})_{ai} = \frac{\sqrt{2}}{F_\pi} \cos\theta_\pi, \tag{B1}$$

and neglecting the terms of and higher than $\mathcal{O}(s/m_{f_i}^2), (i = 2, 3, 4)$ we obtain

$$\sum_{i=2}^{4} g_{f_i \pi \pi} \frac{(p_{\pi_1} + p_{\pi_2})^\mu}{s - m_{f_i}^2 + i m_{f_i} \Gamma_{f_i}(s)} (U_f^{-1})_{ai} = -(p_{\pi_1} + p_{\pi_2})^\mu \left(\frac{\sqrt{2}}{F_\pi} \cos\theta_\pi - \frac{g_{\sigma\pi\pi}}{m_\sigma^2} (U_f^{-1})_{a1}\right). \tag{B2}$$

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Then, we finally arrive at the sum rule for the scalar meson exchanging contribution

$$\sum_{i=1}^{4} g_{f_i\pi\pi} \frac{(p_{\pi_1}+p_{\pi_2})^\mu}{s-m_{f_i}^2+im_{f_i}\Gamma_{f_i}(s)}(U_f^{-1})_{ai} = g_{f_1\pi\pi}\frac{(p_{\pi_1}+p_{\pi_2})^\mu}{s-m_{f_1}^2+im_{f_1}\Gamma_{f_1}(s)}(U_f^{-1})_{a1}$$
$$-(p_{\pi_1}+p_{\pi_2})^\mu \left(\frac{\sqrt{2}}{F_\pi}\cos\theta_\pi - g_{\sigma\pi\pi}\frac{1}{m_\sigma^2}(U_f^{-1})_{a1} \right). \qquad (B3)$$

It should be noticed that the second term of this sum rule arising from the heavier resonance contribution and only the first term leaves once the heavier resonances are neglected from the beginning.

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